Polarized Radiation

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Overview

- Motivation
- Describing Electromagnetic Waves: Amplitude, Intensity $\Rightarrow$ Stokes parameters
- Stokes Parameters $\leftrightarrow$ Polarization
- Measuring Stokes Parameters
- Partial Polarization (Natural Radiation)
- Interaction with Matter
- Summary

Motivation

- Almost all remote sensing uses electromagnetic radiation to sense the remote "object"
- Information on the remote object is "put" into the electromagnetic radiation because of the characteristic scattering, emission and absorption properties of the object
- Scattering, emission and absorption of electromagnetic radiation in the atmosphere or by the earth and ocean surface often shows dependence on polarization

⇒ Polarization is relevant for many remote sensing applications

Describing Electromagnetic Waves: Amplitudes

Monochromatic electromagnetic wave of arbitrary polarization:

$$\mathbf{E}(\mathbf{x},t) = \begin{bmatrix} E_v \\ E_h \end{bmatrix} \mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega t)} = (E_v \mathbf{e}_v + E_h \mathbf{e}_h)\, \mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega t)}$$

where

$\mathbf{E}$ – electric field vector

v,h – denotes vertical and horizontal polarization

E_v, E_h – complex amplitudes, $E_v = a_v \mathrm{e}^{\mathrm{i}\delta_v}$, $E_h = a_h \mathrm{e}^{\mathrm{i}\delta_h}$

$\mathbf{k}$, ω – wavenumber vector, angular frequency

$\begin{bmatrix} E_v \\ E_h \end{bmatrix}$ is also called Jones vector

Physical electric field vector:

$$\tilde{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}[\mathbf{E}(\mathbf{x},t)] = \begin{bmatrix} a_v \cdot \cos(\mathbf{kx} - \omega t + \delta_v) \\ a_h \cdot \cos(\mathbf{kx} - \omega t + \delta_h) \end{bmatrix}$$

Describing Electromagnetic Waves: h? v?

Some words about the terms horizontal and vertical:

- base vectors of horizontal and vertical polarization and the propagation direction ($\mathbf{k}$) are mutually orthogonal
- anything else (even the names “horizontal” and “vertical”!) is a matter of convention
- we use the following one (it does not matter yet):

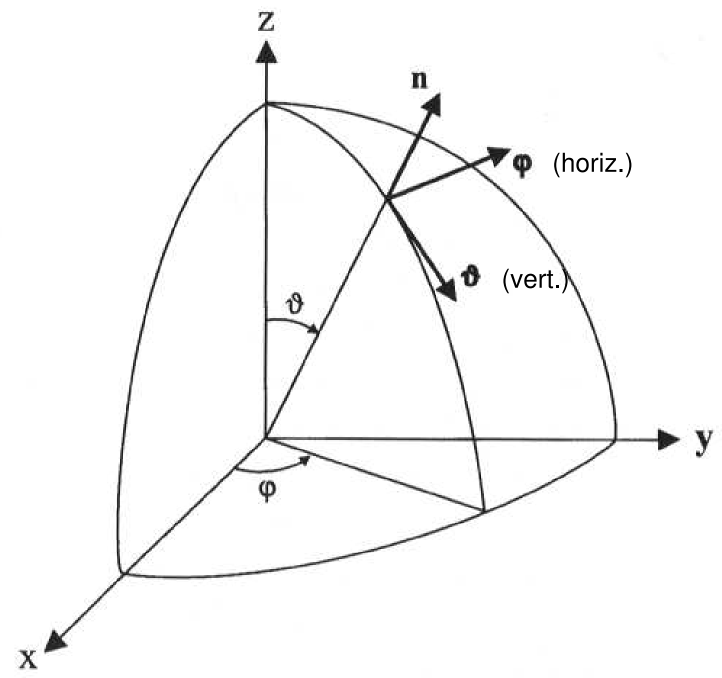


Figure 1 Laboratory coordinate system.

Propagation direction: $\mathbf{n} = \mathbf{k}/k$;
horizontal polarization direction: $\boldsymbol{\phi} = \mathbf{e}_h$;
vertical polarization direction: $\boldsymbol{\theta} = \mathbf{e}_v$

Describing Electromagnetic Radiation: Intensities

- Most instruments don't measure the electric or magnetic field, but rather the time-averaged intensity (energy flux):

$$F = \sqrt{\frac{\epsilon}{\mu}} \overline{(\tilde{\mathbf{E}}(\mathbf{x}, t))^2} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}(E_v E_v^* + E_h E_h^*)$$

- The intensity is not sufficient to fully characterize an electromagnetic wave including polarization. Therefore: define three more intensity quantities, and get the four Stokes Parameters:

Stokes Parameters: Definition

- Commonly, the factor $\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}$ in the flux F is omitted, and we have:

$$
\begin{aligned}
I &= E_v E_v^* + E_h E_h^* \\
Q &= E_v E_v^* - E_h E_h^* \\
U &= -E_v E_h^* - E_h E_v^* \\
V &= i(E_h E_v^* - E_v E_h^*)
\end{aligned}
$$

- Or, using amplitude/phase notation from above:

$$
\begin{aligned}
I &= a_v^2 + a_h^2 \\
Q &= a_v^2 - a_h^2 \\
U &= -a_v a_h \cos(\delta_v - \delta_h) \\
V &= a_v a_h \sin(\delta_v - \delta_h)
\end{aligned}
$$

Stokes Parameters: Some Facts

Stokes parameters

- fully characterize the electromagnetic wave (except for absolute phase)
- are real numbers that can be measured (see below)
- are also denoted S_0, S_1, S_2, S_3 instead of I, Q, U, V.
- Written as row or column vector: Stokes Vector
- are sometimes defined with different signs and normalizations
- in particular: Stokes parameters can be normalized to represent radiance or irradiance
- important variant: modified Stokes parameters: (I_v, I_h, U, V), where $I_v = E_v E_v^*$ and $I_h = E_h E_h^*$

Meaning of Stokes Parameters: Polarization Ellipse (1)

- To get the relation between Stokes parameters and the polarization, look at the curve that the tip of the physical electric field vector describes with time at a fixed position $\mathbf{x_0}$:

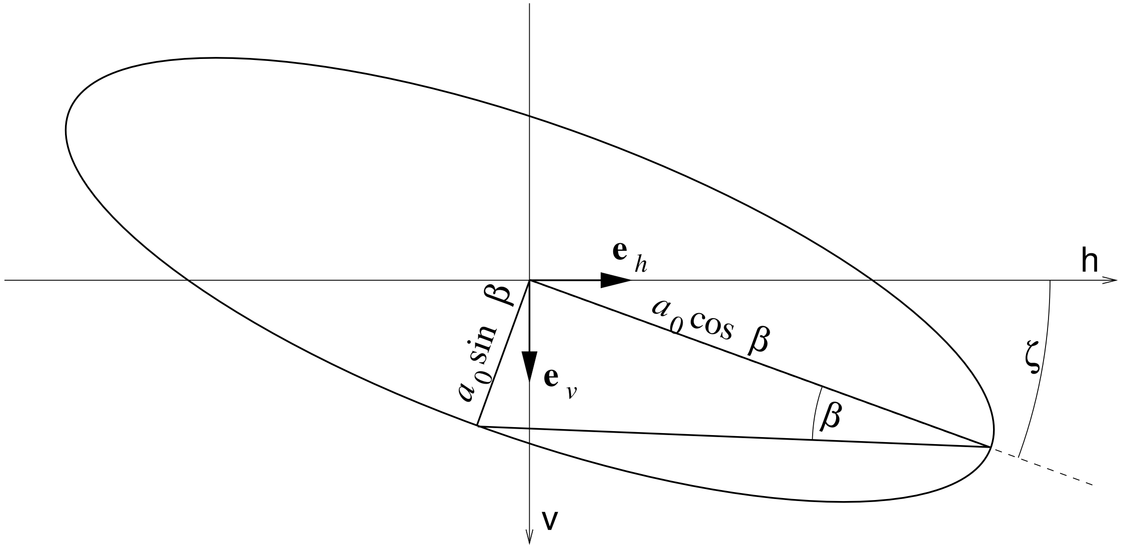
$$\begin{aligned}\tilde{E}_v(t) &= a_v \cos(\Delta_v - \omega t) \\ \tilde{E}_h(t) &= a_h \cos(\Delta_h - \omega t)\end{aligned}$$

  where $\Delta_{v,h} = \mathbf{k x_0} + \delta_{v,h}$.
- Should in general be an ellipse of arbitrary orientation and shape (polarization ellipse), so we want to rewrite as:

$$\begin{aligned}\tilde{E}_v(t) &= a_0(\sin\beta\cos(\omega t)\cos\zeta + \cos\beta\sin(\omega t)\sin\zeta) \\ \tilde{E}_h(t) &= a_0(-\sin\beta\cos(\omega t)\sin\zeta + \cos\beta\sin(\omega t)\cos\zeta)\end{aligned}$$

Meaning of Stokes Parameters: Pol. Ellipse (2)

$$\tilde{E}_v(t) = a_0(\sin\beta\cos(\omega t)\cos\zeta + \cos\beta\sin(\omega t)\sin\zeta)$$
$$\tilde{E}_h(t) = a_0(-\sin\beta\cos(\omega t)\sin\zeta + \cos\beta\sin(\omega t)\cos\zeta)$$



- β – ellipticity angle ($|\tan\beta|$ = ratio of semi-minor and semi-major axes = ellipticity)
- $\beta \in [-45^\circ, +45^\circ]$, clockwise rotation for $\beta < 0$
- ζ – orientation angle (angle of major axis with h axis), $\zeta \in [0^\circ, 180^\circ]$

Meaning of Stokes Parameters: Pol. Ellipse (3)

- Relation between

orientation and shape of ellipse (ζ, β)

and

amplitude/phase of horizontal and vertical component (a_v, a_h; δ_v, δ_h)

(*established by tedious but elementary calculation, omitted here*):

$$\begin{aligned} \frac{a_v a_h}{a_0^2}\cos(\delta_v - \delta_h) &= \tfrac{1}{2}\sin(2\zeta)\cos(2\beta) \\ -\frac{a_v a_h}{a_0^2}\sin(\delta_v - \delta_h) &= \tfrac{1}{2}\sin(2\beta) \\ a_v^2 - a_h^2 &= -a_0^2\cos(2\zeta)\cos(2\beta) \end{aligned}$$

where $a_0^2 = a_v^2 + a_h^2$

- not very practical, but can be used to rewrite Stokes parameters:

Meaning of Stokes Parameters: Polarization

$$
\begin{aligned}
I &= a_0^2 \\
Q &= -a_0^2 \cos(2\zeta)\cos(2\beta) \\
U &= -a_0^2 \sin(2\zeta)\cos(2\beta) \\
V &= -a_0^2 \sin(2\beta)
\end{aligned}
$$

- Orientation angle ζ from: $\tan(2\zeta) = \frac{U}{V}$ (and $\cos(2\zeta)$ must have sign of $-Q$)
- Ellipticity angle β from: $\tan(2\beta) = -\frac{V}{(Q^2+U^2)^{1/2}}$
- Stokes parameters not totally independent:

$I^2 = Q^2 + U^2 + V^2$ applies for plane monochromatic waves!

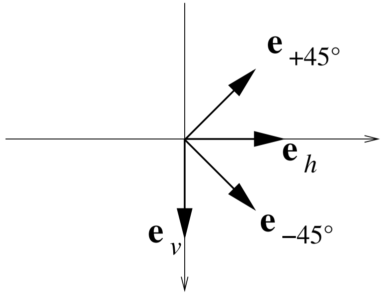
Measuring Stokes Parameters: I, Q

- I is the total intensity, can be measured "easily" (radiometer)
- $Q = I_v - I_h$, where $I_v = E_v E_v^* = a_v^2$ (I_h analogous)
- thus, Q can be measured with the aid of polarizers (optical regime) or oriented dipole antennas (microwave regime).
- U and V can also be expressed as intensity differences, but that requires some additional consideration:

Measuring Stokes Parameters: U

- Define a different linear basis:

$$\mathbf{e}_{+45^\circ} = \sqrt{\tfrac{1}{2}}\,(\mathbf{e}_h - \mathbf{e}_v)$$
$$\mathbf{e}_{-45^\circ} = \sqrt{\tfrac{1}{2}}\,(\mathbf{e}_h + \mathbf{e}_v)$$



- Rewrite $\mathbf{E}$ (omitting the oscillatory factor)

$$\mathbf{E} = (E_v\mathbf{e}_v + E_h\mathbf{e}_h)$$
$$\mathbf{E} = \underbrace{\sqrt{\tfrac{1}{2}}\,(E_v + E_h)}_{E_{-45^\circ}}\,\mathbf{e}_{-45^\circ} + \underbrace{\sqrt{\tfrac{1}{2}}\,(-E_v + E_h)}_{E_{+45^\circ}}\,\mathbf{e}_{+45^\circ}$$

Measuring Stokes Parameters: U (ctd.)

- Defining intensities of the two components:

$$I_{\pm 45^\circ} = E_{\pm 45^\circ} E^*_{\pm 45^\circ}$$

we get:

$$\begin{aligned} I_{+45^\circ} - I_{-45^\circ} &= \tfrac{1}{2}(-E_v + E_h)(-E_v^* + E_h^*) - \tfrac{1}{2}(E_v + E_h)(E_v^* + E_h^*) \\ &= -E_v E_h^* - E_h E_v^* \\ &= U \end{aligned}$$

- The two intensities I_{+45° and I_{-45° can also be measured with the aid of polarizers or oriented antennas

Measuring Stokes Parameters: V

- In a similar way, we rewrite $\mathbf{E}$ in the circular basis, with the basis vectors:

$$\begin{aligned}\mathbf{e}_{LH} &= \sqrt{\tfrac{1}{2}}\,(\mathbf{e}_v + i\mathbf{e}_h)\\ \mathbf{e}_{RH} &= \sqrt{\tfrac{1}{2}}\,(\mathbf{e}_v - i\mathbf{e}_h)\end{aligned}$$

$$\mathbf{E} = \underbrace{\sqrt{\tfrac{1}{2}}\,(E_v - iE_h)}_{E_{LH}}\,\mathbf{e}_{LH} + \underbrace{\sqrt{\tfrac{1}{2}}\,(E_v + iE_h)}_{E_{RH}}\,\mathbf{e}_{RH}$$

- Intensity difference

$$\begin{aligned}I_{RH} - I_{LH} &= \tfrac{1}{2}(E_v + iE_h)(E_v^* - iE_h^*) - \tfrac{1}{2}(E_v - iE_h)(E_v^* + iE_h^*)\\ &= i(E_h E_v^* - E_v E_h^*)\\ &= V\end{aligned}$$

Measuring Stokes Parameters: V (ctd.)

- In microwave regime, I_{RH} and I_{LH} can be measured with appropriate helical antennas,
- In optical regime, no direct measurement possible, but:
- combination of a quarter wave plate and a polarizer, aligned at -45° with respect to the h-axis, yields:

$$\mathbf{E}'' = \sqrt{\tfrac{1}{2}}\,(E_v + iE_h)\,\mathbf{e}_{-45^\circ}$$

- and measuring the intensity now:

$$\begin{aligned} I'' &= \left|\mathbf{E}''\right|^2 \\ &= \tfrac{1}{2}\,(E_v + iE_h)\,(E_v^* - iE_h^*) \\ &= \tfrac{1}{2}\left(|E_v|^2 + |E_h|^2 - i(E_v E_h^* - E_h E_v^*)\right) \\ &= \tfrac{1}{2}(I + V) \end{aligned}$$

Measuring Stokes Parameters: Summary

Stokes parameters in terms of intensities of orthogonal components:

$$\begin{aligned} I &= I_v + I_h = I_{-45^\circ} + I_{+45^\circ} = I_{RH} + I_{LH} \\ Q &= I_v - I_h \\ U &= I_{+45^\circ} - I_{-45^\circ} \\ V &= I_{RH} - I_{LH} \end{aligned}$$

- I: total intensity
- Q and U: both related to linear polarization
- V is related to circular polarization
- e.g.: Horizontal polarization: $(I, Q, U, V) = (I, -I, 0, 0)$
- e.g.: $+45^\circ$ linear polarization: $(I, Q, U, V) = (I, 0, I, 0)$
- e.g.: Right-circular polarization: $(I, Q, U, V) = (I, 0, 0, I)$

The Real World: Partial Polarization

- All the above valid for the ideal case: plane monochromatic waves, i.e., amplitudes (a_v, a_h) and phases (δ_v and δ_v) fixed and constant, i.e., emitted by one coherent source
- In reality: Natural radiation usually from many incoherent (in space and time) emission events
- thus, amplitudes and phases fluctuate, time scale $\gg$ wave period, but $<$ integration time of instrument.
- Since the emission events are incoherent, their individual Stokes parameters (intensities) can be summed (and averaged over time), and an instrument measures this sum and average:

$$I = \sum_i I_i \ , \ \ Q = \sum_i Q_i \ , \ \ U = \sum_i U_i \ , \ \ V = \sum_i V_i$$

- If the fluctuations are random (uniformly distributed), Q, U and V cancel; the radiation is unpolarized
- If Q, U and V are non-zero, we speak of partial polarization
- Partial polarization cannot be expressed conveniently by the electric field vector!

Partial Polarization Quantified

- For each i (one emission event): $I_i^2 = Q_i^2 + U_i^2 + V_i^2$
- Thus:

$$I^2 \geq Q^2 + U^2 + V^2$$

  (proof upon request)

- Define a degree of polarization:

$$p = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

Partially Polarized $=$ Polarized $+$ Unpolarized

- Define intensity of polarized component: $I_p^2 = Q^2 + U^2 + V^2$
- ...and intensity of unpolarized component: $I_u = I - I_p$
- Thus, partially polarized radiation can be regarded as as superposition of completely polarized radiation (I_p, Q, U, V) and unpolarized radiation $(I_u, 0, 0, 0)$
- Define degree of linear polarization: $p_{lin} = \frac{\sqrt{Q^2+U^2}}{I}$
- ...and degree of circular polarization: $p_{circ} = \frac{V}{I}$

Polarization of Radiation in the Atmosphere

- Radiation from the sun is unpolarized
- Radiation emitted from the ground can be weakly polarized, dependent on material, texture, direction
- Radiation emitted by the atmosphere is almost unpolarized because of the random orientation of air molecules (exception: Zeemann effect of oxygen induced by Earth's magnetic field)
- Scattering of radiation by oriented particles (e.g. cirrus clouds) is sensitive to polarization, increases degree of polarization
- Typically $I > |Q| > |U|, |V|$
- Note: $Q, U, V < 0$ possible $\Rightarrow$ brightness temperature??

Interaction with Matter (Scattering): Amplitude Matrix

- Going back to the fields and amplitudes (Jones formalism)
- Scattering by particles (or surfaces): Amplitude and phase of the electromagnetic wave change.
- Incident: plane wave, scattered (far field): spherical wave,

$$\begin{bmatrix} E_{v'}^{\mathrm{sca}} \\ E_{h'}^{\mathrm{sca}} \end{bmatrix} = \frac{\exp(\mathrm{i}kR)}{R} \mathbf{S}(\mathbf{n}, \mathbf{n}', \ldots) \begin{bmatrix} E_{v}^{\mathrm{inc}} \\ E_{h}^{\mathrm{inc}} \end{bmatrix}$$

where

v, h – horiz., vert. polarization direction for incident wave
$\mathbf{n}$ – propagation direction of incident wave (equiv.: θ, ϕ)
v', h' – horiz., vert. polarization direction for scattered wave
$\mathbf{n}'$ – propagation direction of scattered wave (equiv.: θ', ϕ')
$\mathbf{S}$ – complex 2 by 2 matrix, called [scattering] amplitude matrix, also scattering function matrix or Jones matrix, depends on both incident and scattered direction, on the scatterer (shape, orientation, dielectric properties) and thus also on the wavelength

Interaction with Matter: Amplitude Matrix Examples

Amplitude matrix for some optical devices (forward “scattering”, i.e., $\mathbf{n} = \mathbf{n}'$):

- Linear polarizer, aligned with v-direction:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- 90° linear retarder (quarter wave plate), fast axis along v:

$$\mathrm{e}^{-\mathrm{i}q} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi/4} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\pi/4} \end{bmatrix}$$

where q is some material constant of the retarder (just an absolute phase)

Interaction with Matter: Scattering Matrix

- Transition from Jones vectors to Stokes vectors (quite straightforward, using the definition of I, Q, U, V in terms of E_v and E_h):

$$\mathbf{I}^{\text{sca}} = \frac{1}{R^2}\mathbf{Z}(\mathbf{n}, \mathbf{n}', \ldots)\mathbf{I}^{\text{inc}}$$

  where $\mathbf{I}^{\text{sca}}$ and $\mathbf{I}^{\text{inc}}$ are the Stokes vectors of the scattered and incident radiation.
- $\mathbf{Z}$: real 4 by 4 matrix with many names: scattering matrix, Stokes matrix, Mueller matrix, phase matrix, depending on the context and/or book (and, of course, they use various letters, typically $\mathbf{S}$, $\mathbf{M}$, $\mathbf{Z}$).

$$\begin{aligned}
Z_{11} &= \frac{1}{2}(|S_{11}|^2 + |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2) \\
Z_{12} &= \frac{1}{2}(|S_{11}|^2 - |S_{12}|^2 + |S_{21}|^2 - |S_{22}|^2) \\
Z_{13} &= -\mathrm{Re}(S_{11}S_{12}^* + S_{22}S_{21}^*) \\
Z_{14} &= -\mathrm{Im}(S_{11}S_{12}^* - S_{22}S_{21}^*) \\
&[\ldots] \\
Z_{44} &= \mathrm{Re}(S_{22}S_{11}^* - S_{12}S_{21}^*)
\end{aligned}$$

Interaction with Matter: Scattering Matrix Examples

- Linear polarizer, aligned with v direction:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- 90° linear retarder (quarter wave plate), fast axis along v:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Caution: Beware of Difference of Geometry!

Amplitude matrix and scattering matrix strongly depend on definition of polarization directions with respect to incident and scattering directions. Two main systems:

1. [Used Here] Horizontal and vertical polarization defined with respect to propagation direction and z-axis; independently for incident and scattering direction [*Tsang, Mishchenko*].
   $\mathbf{S}(\theta, \phi, \theta', \phi')$

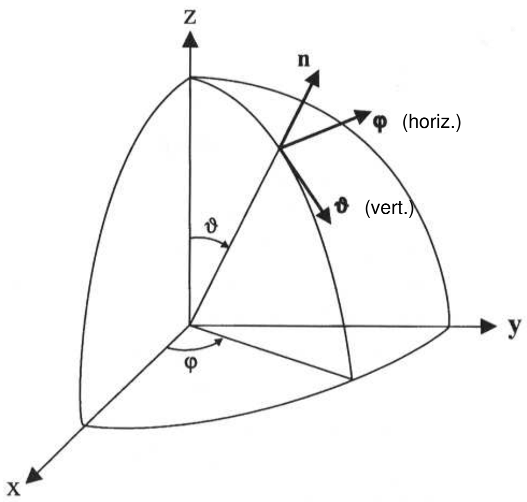


Figure 1 Laboratory coordinate system.

Useful if the scatterers can have various orientations and if the incident radiation can come from various directions (e.g. RT in atmosphere with scattering by cirrus clouds $\longrightarrow$ ARTS)

Caution: Beware of D.o.G.! (ctd.)

2. Incident direction fixed along z-axis, polarization directions defined with respect to scattering plane (plane containing the z-axis and the scattering direction) [*Bohren/Huffman*, *Chandrasekhar*, *van de Hulst*]; can be seen as a special case of the previous configuration
   $\mathbf{S}(\Theta, \Phi)$, Θ = "scattering angle"

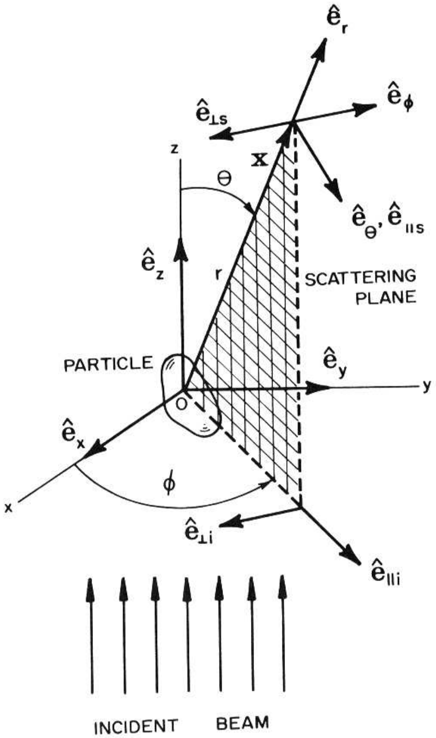


Figure 3.3 Scattering by an arbitrary particle.

Polarization directions: $\mathbf{e}_\perp$, $\mathbf{e}_\parallel$

$\mathbf{e}_{\perp s} = \mathbf{e}_{\perp i}$; $\mathbf{e}_{\parallel s} \neq \mathbf{e}_{\parallel i}$;

Useful if one scatterer is considered and/or the direction of incident radiation is fixed.

Note also different definition of amplitude matrix:

$$\begin{bmatrix} E_{v'}^{\mathrm{sca}} \\ E_{h'}^{\mathrm{sca}} \end{bmatrix} = \frac{\exp(\mathrm{i}kR)}{-\mathrm{i}kR} \mathbf{S}(\Theta, \Phi, \ldots) \begin{bmatrix} E_v^{\mathrm{inc}} \\ E_h^{\mathrm{inc}} \end{bmatrix}$$

Summary

- Electric field vector or Stokes parameters both fully characterize electromagnetic radiation
- Stokes parameters are intensity quantities: Can be measured more readily than the electric field
- Natural radiation is a superposition of radiation from many incoherent sources $\Rightarrow$ Polarization of all these contributions partially or fully cancels
- Partial polarization conveniently described by Stokes parameters
- Typically: $I > |Q| > |U|, |V|$
- Interaction with matter: complex 2×2 amplitude matrix (for electric field)
- Interaction with matter: real 4×4 scattering matrix (for Stokes parameters)
- Beware of different conventions for the geometry, polarization directions, definition of Stokes parameters etc.

References I found useful

- D.S. Kliger, J.W. Lewis, C.E. Randall: *Polarized Light in Optics and Spectroscopy*, Academic Press, 1990. (Ch. 1 to 5)
- S. Chandrasekhar: *Radiative Transfer*, Dover Publ., 1960. (§15)
- J. Stone: *Radiation and Optics*, McGraw-Hill, 1963.
- C.E. Bohren, D.R. Huffman: *Absorption and Scattering of Light by Small Particles*, Wiley, 1998. (Ch. 2 and 3)
- M.I. Mishchenko, J.W. Hovenier, L.D. Travis: *Light Scattering by Nonspherical Particles*, Academic Press, 2000. (Ch. 1 and 3)
- B.T. Draine, P.J. Flatau: *User Guide for the Discrete Dipole Approximation Code DDSCAT (Version 5a10)*, http://arxiv.org/abs/astro-ph/0008151v4. (Ch. 20 to 22)

Just in Case: Proof of $I^2 \geq Q^2 + U^2 + V^2$

Go back to the amplitude/phase notation (cf. *Chandrasekhar*, 1960; chap. I.15):

$$I = \sum_i I_i = \sum_i \left(a_v^{(i)}\right)^2 + \sum_i \left(a_h^{(i)}\right)^2 \quad (1)$$

$$Q = \sum_i Q_i = \sum_i \left(a_v^{(i)}\right)^2 - \sum_i \left(a_h^{(i)}\right)^2 \quad (2)$$

$$U = \sum_i U_i = -2 \sum_i a_v^{(i)} a_h^{(i)} \cos \delta^{(i)} \quad (3)$$

$$V = \sum_i V_i = 2 \sum_i a_v^{(i)} a_h^{(i)} \sin \delta^{(i)} \quad (4)$$

where $\delta^{(i)} = \delta_v^{(i)} - \delta_h^{(i)}$.

We get

$$
\begin{aligned}
I^2 - Q^2 - U^2 - V^2 \quad = \quad & 4 \sum_i \left(a_v^{(i)} \right)^2 \sum_i \left(a_h^{(i)} \right)^2 \\
& -4 \left(\sum_i a_v^{(i)} a_h^{(i)} \cos \delta^{(i)} \right)^2 \\
& -4 \left(\sum_i a_v^{(i)} a_h^{(i)} \sin \delta^{(i)} \right)^2
\end{aligned} \tag{5}
$$

First term on right-hand side can be rearranged as

$$
\sum_i \left(a_v^{(i)} a_h^{(i)} \right)^2 + \sum_{\substack{i,j \\ i \neq j}} \left(a_v^{(i)} a_h^{(j)} \right)^2 \tag{6}
$$

Rearrange the other two terms similarly to yield

$$
- \sum_i \left(a_v^{(i)} a_h^{(i)} \right)^2 \left[\cos^2 \delta^{(i)} + \sin^2 \delta^{(i)} \right] \tag{7}
$$

$$-\sum_{\substack{i,j\\ i\neq j}} a_v^{(i)} a_h^{(i)} a_v^{(j)} a_h^{(j)} \left[\cos\delta^{(i)}\cos\delta^{(j)} + \sin\delta^{(i)}\sin\delta^{(j)}\right]$$

Putting this into Eq. 5 (and dividing by 4), the sums over just i cancel and we get:

$$\begin{aligned}(I^2 - Q^2 - U^2 - V^2)/4 &= \sum_{\substack{i,j\\ i\neq j}} \left(a_v^{(i)} a_h^{(j)}\right)^2 \\ &\quad - \sum_{\substack{i,j\\ i\neq j}} a_v^{(i)} a_h^{(i)} a_v^{(j)} a_h^{(j)} \cos(\delta^{(i)} - \delta^{(j)})\end{aligned} \tag{8}$$

where the cosine addition theorem was used. In the summation, we now change from $i \neq j$ to $i < j$, so we have to symmetrize the first term (the second term is already symmetric with respect to i and j and therefore just gets a factor 2):

$$\begin{aligned}(I^2 - Q^2 - U^2 - V^2)/4 &= \sum_{\substack{i,j\\ i<j}} \left[\left(a_v^{(i)} a_h^{(j)}\right)^2 + \left(a_v^{(j)} a_h^{(i)}\right)^2\right. \\ &\quad \left. -2\left(a_v^{(i)} a_h^{(j)}\right)\left(a_v^{(j)} a_h^{(i)}\right)\cos(\delta^{(i)} - \delta^{(j)})\right]\end{aligned} \tag{9}$$

Each summand of the sum on the right-hand side is positive, since it is greater than or equal to $(a_v^{(i)} a_h^{(j)} - a_v^{(j)} a_h^{(i)})^2$, which completes the proof.

The right-hand side vanishes only if $\delta^{(i)} = \delta^{(j)}$ and $a_v^{(i)}/a_h^{(i)} = a_v^{(j)}/a_h^{(j)}$ for all i, j, i.e., if the phase difference and amplitude ratio between the horizontal and vertical component of the electric field is the same for all contributions, in other words: if all contributions have the same polarization.