1.1
Lecture 1.1  Introduction

Maxwell-Boltzmann to Alder & Wainwright via long-tailed probability distributions and the scaling of atmospheric temperature and winds.
Key References

[emergence of fluid flow from molecular dynamics]

[generalized scale invariance, statistical multifractals]


[most recent review of generalized scale invariance]

The correspondence and coupling of the microscopic and macroscopic processes in the atmosphere.
Maxwellian Velocity Distribution

\[ \frac{N_v (v)}{N} = 4\pi \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} v^2 e^{-\frac{KE}{kT}} \]

- \( v_p \) = Most Probable Speed = \( \sqrt{\frac{2kT}{m}} \)
- \( v_{av} \) = Average Speed = \( \sqrt{\frac{8kT}{\pi m}} \)
- \( v_{rms} \) = Root-Mean-Square Speed = \( \sqrt{\frac{3kT}{m}} \)
Maxwellian speed PDFs, $m = 28$: $T$ dependence
Maxwellian speed PDFs: mass dependence

![Graph showing Maxwellian speed PDFs for different gases at T = 25°C. The graph plots dN/dv against speed (m/s) for Xenon, Carbon Dioxide, Nitrogen, Water, Helium, and Hydrogen. Each gas has a distinct curve representing its probability density function.](image)
DEFINITION

Atmospheric temperature is what is measured by your thermometer (meteorological).

The absolute temperature, $T$, of a system is the reciprocal of the derivative of the entropy, $S$, with respect to its energy, $E$:

$$\frac{dS}{dE} = \frac{1}{kT}$$

[Landau & Lifshitz, (1980), Statistical Physics, Course of Theoretical Physics, Vol. 5, 3rd ed., Chapters 1 - 3.]

$T$ is purely statistical, having strict meaning only for macroscopic bodies at equilibrium. One can of course observe with a calibrated thermometer. It averages over the velocity distribution of the molecules impinging upon it.
MICROSCOPIC VIEW OF TEMPERATURE

- The total kinetic energy of $N$ classical particles is $3NkT/2$.

- In terms of distributions of molecular velocity $v$:

$$
\overline{v_x^2} = \sqrt{\frac{m}{2\pi kT}} \int_{-\infty}^{\infty} v_x^2 e^{-m v_x^2 / 2kT} dv_x = \frac{kT}{m}
$$

Slide 11
Scale Invariance and Statistical Multifractality

$$S_q(r; \Psi) = \langle | \Psi(x + r) - \Psi(x) |^q \rangle$$

$q^\text{th}$ order structure function $S$ of variable $\Psi(x)$

If a plot of $\log S_q$ vs. $\log(r)$ is linear with slope $\zeta(q)$, then $\zeta(q)$ is a scaling exponent for $\Psi(x)$, which therefore has scale invariance and power law PDFs.

Define

$$H(q) = \frac{\zeta(q)}{q}$$

Further define

$$H = H_q + \frac{K(q)}{q}$$

*We will be interested in $K(q)$ and $H$*
To obtain $K(q)$, consider $\Psi(x)$ to have been observed at finite intervals $x = 1, 2, 3, \ldots, x_{\max}$ and define:

$$\varepsilon(1, x) = \frac{\Psi(x + 1) - \Psi(x)}{\langle \Psi(x + 1) - \Psi(x) \rangle}$$

for $x = 1, 2, 3, \ldots, x_{\max}$

$$\varepsilon(r, x) = \frac{1}{r} \sum_{j=x}^{x+r-1} \varepsilon(1, j)$$

for $x = 1, 2, 3, \ldots, x_{\max} - r$

then a plot of $\log \langle \varepsilon(r, x) \rangle^q$ vs $\log r$ has slope $-K(q)$ and a plot of $-K(q)$ vs $q$ shows a convex function with $K(0) = K(1) = 0$.

We can now note equivalences between scale invariance and statistical thermodynamics.
Formal equivalences between scale invariant (r.h.s.) and statistical thermodynamic (l.h.s.) variables

\[ T = \frac{1}{qk_{\text{Boltzmann}}} \quad \text{temperature} \]

\[ f = e^{-K(q)} \quad \text{partition function} \]

\[ G = -\frac{K(q)}{q} \quad \text{Gibbs free energy} \]

This offers possible links:

- molecular scale
- statistical thermodynamics
- macroscopic scale invariant observables
Basic scaling relations for atmospheric turbulence

Consider horizontal velocity $v$ with fluctuations $\Delta v$ over horizontal length interval $\Delta x$ and vertical interval $\Delta z$:

$$\Delta v = \varnothing(\text{horiz}).(\Delta x)^{H(\text{horiz})}$$

$$\Delta v = \varnothing(\text{vert}).(\Delta z)^{H(\text{vert})}$$

where $\varnothing(\text{horiz})$ is the turbulent energy flux $\varepsilon$ in the vertical and $\varnothing(\text{vert})$ is the turbulent energy flux; $H(\text{horiz})$ and $H(\text{vert})$ are the associated power law exponents. The Kolmogorov law for isotropic turbulence is obtained by setting

$$\varnothing(\text{horiz}) = \varnothing(\text{vert}) = \varepsilon^{1/3} \quad \text{and} \quad H(\text{horiz}) = H(\text{vert}) = 1/3$$

However, as we shall see later, this is at odds with observations. We therefore assume via dimensional analysis relations that preserve scaling while permitting anisotropy:

$$\varnothing(\text{horiz}) = \varepsilon^{1/3} \quad \text{implying} \quad H(\text{horiz}) = 1/3$$

$$\varnothing(\text{vert}) = \eta^{1/5} \quad \text{implying} \quad H(\text{vert}) = 3/5$$

where energy flux $\varepsilon \ (m^2s^{-3})$ dominates in the horizontal and buoyancy variance flux $\eta \ (m^2s^{-5})$ dominates in the vertical.
Alder & Wainwright (1970): *molecular dynamics* simulation of a flux applied to an equilibrated Maxwellian population results in the emergence of vortices on scales of $10^{-12}$ seconds & $10^{-8}$ metres.
NOAA-NCEP GFS 0.5° RESOLUTION ANALYSIS, FLECHES & ISOTACHS

Isotachs (m/s) 2 mb Fri 12Z 06Jul2007
1.1 Summary

- Organized flow emerges from a randomized (thermal) population of Maxwellian ‘billiard balls’ subject to an anisotropic flux simulated by molecular dynamics, on very short time and space scales.

- Observed temperature distributions in the atmosphere are not of Maxwell-Boltzmann character.

- To accommodate the observed non-M-B PDFs, we must lift the assumption of isotropy in the atmosphere.

- The atmosphere has permanent anisotropies: gravity, planetary rotation, solar beam, planetary surface and, for any one scale except the great circle, larger scale winds.

- Abandoning isotropic motions on any scale yields predictions:

  \[ H(\text{horiz}) = \frac{1}{3} \]
  \[ H(\text{vert}) = \frac{3}{5} \]

  and so \[ \frac{H(\text{horiz})}{H(\text{vert})} = \frac{5}{9} \]

*How do these work out in practice? See next lecture ……