Statistical properties of light reflected and transmitted by a thick horizontally inhomogeneous turbid layer

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The paper is devoted to the introduction of simple analytical relationships between statistical distributions of various radiative transfer characteristics for an inhomogeneous turbid layer with the extinction coefficient varying in the horizontal direction. Results are valid for an optically thick light-scattering layer having arbitrary local scattering laws and single-scattering albedos. It is shown that the statistical distribution of the optical thickness can be obtained directly from the measured statistical distribution of the reflectance or transmittance of a horizontally inhomogeneous light-scattering layer. © 2005 Optical Society of America

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1. INTRODUCTION

Radiative transfer characteristics of horizontally inhomogeneous media (e.g., clouds and snow) can be studied using Monte Carlo numerical calculations.[1,2] This technique allows one to obtain accurate results for arbitrary spatial distributions of scatterers. However, for such calculations to be exact, a lot of computational time for optically thick media (especially at wavelengths where light absorption is weak) is required. Also, simple physics behind relationships between physical parameters under study is hidden in numerical results.

Kokhanovsky[3] established relationships between various radiative transfer characteristics of horizontally inhomogeneous turbid media (e.g., the average diffuse transmittance and the average bidirectional reflection function) in the framework of the asymptotic theory as given in Refs. 4 and 5. The results obtained are valid for optically thick weakly absorbing media only.

The task of this work is to extend the study of Kokhanovsky[3] to the case of arbitrary absorption. Another difference from the previous work is that direct relationships between statistical distributions of various optical characteristics are derived in the case of thick turbid layers. The previous work was concentrated on average characteristics and coefficients of variances.

The limitation with respect to the optical thickness τ of a turbid layer remains. So the technique cannot be applied to the case of optically thin turbid layers.

2. OPTICAL PROPERTIES OF INHOMOGENEOUS SCATTERING MEDIA OUTSIDE ABSORPTION BANDS

We assume that scatterers are confined in a plane-parallel slab illuminated from above by an infinitely wide uniform light beam (e.g., clouds illuminated by the Sun). The angular distribution of light scattered by an elementary volume of a turbid medium is assumed to be constant at any given place inside the medium under study. Effects of light absorption are neglected, which is an accurate assumption in the spectral regions positioned far from absorption bands of substances present in the scattering layer under study. However, we assume that the extinction coefficient fluctuates in the horizontal direction (e.g., owing to fluctuations in the concentration of scatterers). Such fluctuations lead to fluctuations of the optical thickness. Strictly speaking, this makes one-dimensional (1-D) radiative transfer theory[6] (RTT) not applicable to the case under study.

However, a slight modification of the theory allows us to extend the applicability of the 1-D RTT to treat (although approximately) this more complex case. In particular, it is assumed that the reflection function and also other radiative characteristics of a slab can be presented as integrals:

$$\bar{Y} = \int_0^\infty Y(\tau) f_\tau(\tau) \mathrm{d}\tau, \tag{1}$$

where Y is a radiative transfer characteristic under study and $f_\tau(\tau)$ is the statistical distribution of the optical thickness τ. This approximation is called the independent pixel approximation. The accuracy of this approximation as compared with Monte Carlo calculations was studied in Refs. 2, 7, and 8. Barker[9] proposed the use of the gamma distribution

$$f_\tau(\tau) = \frac{\mu^{\mu+1}}{\tau_0^{\mu+1}\Gamma(\mu+1)} \tau^\mu \exp\left(-\mu\frac{\tau}{\tau_0}\right) \tag{2}$$

in conjunction with the two-stream approximations[10] for the characteristics $Y(\tau)$ to account for the horizontal inhomogeneity effects of turbid layers. Then the integral in Eq. (1) can be evaluated analytically. The statistical distribution given by Eq. (2) is shown in Fig. 1 at $\mu = 8$ (the coefficient of variance equals 1/3) and $\tau_0 = 20$.

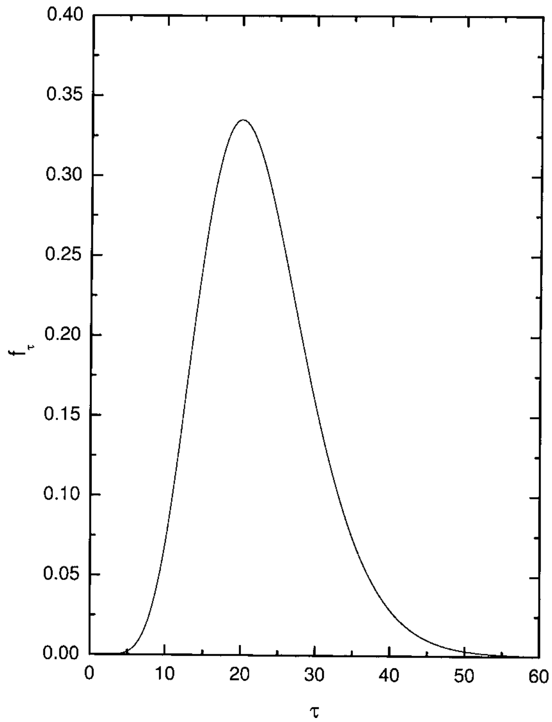

Fig. 1. Statistical distribution of the optical thickness calculated using Eq. (2) at $\tau_0=20$ and $\mu=8$.

Such a technique leads to simple analytical equations for a number of radiative transfer characteristics important in global circulation models. However, the two-stream approximation cannot be used for the bidirectional reflectance and transmittance calculations. Then the asymptotic RTT can be used.[3] It follows in the framework of this theory[5,11,12] for a homogeneous optically thick nonabsorbing plane-parallel turbid layer that

$$R(\xi,\eta,\varphi)=R_{0\infty}(\xi,\eta,\varphi)-tK_0(\xi)K_0(\eta), \tag{3}$$

$$T(\xi,\eta)=tK_0(\xi)K_0(\eta), \tag{4}$$

where $R(\xi,\eta,\varphi)$ is the reflection function, $T(\xi,\eta)$ is the transmission function, $R_{0\infty}(\xi,\eta,\varphi)$ is the reflection function of a semi-infinite nonabsorbing medium having the same optical properties as a finite layer under study, $K_0(\xi)$ is the escape function for a nonabsorbing layer determined via $R_{0\infty}(\xi,\eta,\varphi)$ as[11]

$$K_0(\xi)=\frac{3}{4}\left\{\xi+\frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_0^1\mathrm{d}\eta R_{0\infty}(\xi,\eta,\varphi)\eta^2\right\}. \tag{5}$$

ξ is the cosine of the zenith illumination angle and η is the cosine of the zenith observation angle, φ is the relative azimuth, and

$$t=\frac{1}{a+b\tau} \tag{6}$$

is the global transmittance defined as

$$t=\frac{2}{\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_0^1\xi\mathrm{d}\xi\int_0^1\eta\mathrm{d}\eta T(\xi,\eta,\varphi), \tag{7}$$

and we accounted for the fact that the light diffused through thick layers is azimuthally independent [see Eq. (4)]. The parameters a and b are given as follows[13]: $a=1.07$ and $b=0.75(1-g)$ for arbitrary phase functions. Here g is the asymmetry parameter. This parameter is derived from Mie theory in the case of spherical scatterers.[14] For instance, its value is close to 0.85 for water clouds in the visible.[13]

The escape function is normalized as follows[11]:

$$2\int_0^1 K_0(\eta)\eta\mathrm{d}\eta=1. \tag{8}$$

Sobolev[11] proposed the following approximation for the escape function valid at arbitrary phase functions and $\eta>0.2$: $K_0(\eta)=3(1+2\eta)/7$. This relationship allows one to derive simple equations for the diffuse transmittance,

$$t_d(\xi)=2\int_0^1\bar{T}(\xi,\eta)\eta\mathrm{d}\eta, \tag{9}$$

plane albedo,

$$r_d(\xi)=2\int_0^1\bar{R}(\xi,\eta)\eta\mathrm{d}\eta, \tag{10}$$

and the spherical albedo,

$$r=2\int_0^1 r_d(\xi)\xi\mathrm{d}\xi, \tag{11}$$

where

$$\bar{R}(\xi,\eta)=\frac{1}{2\pi}\int_0^{2\pi}R(\xi,\eta,\varphi)\mathrm{d}\varphi \tag{12}$$

and similar for $\bar{T}$. So we have[13]

$$t_d(\xi)=tK_0(\xi),\qquad r_d(\xi)=1-tK_0(\xi),\qquad r=1-t, \tag{13}$$

where we accounted for the fact that $r_{d\infty}=1$, $r_\infty=1$ by definition for nonabsorbing semi-infinite media.

Radiative characteristics r, t, $r_d(\xi)$, $t_d(\xi)$, $R(\xi,\eta,\varphi)$, and $T(\xi,\eta)$ can be measured. In particular, pyranometers installed on an aircraft are used to measure $r_d(\xi)$ over a cloud field.

We pose the following question. Is it possible to obtain the statistical distribution of the optical thickness $f_\tau(\tau)$ from the measurements of the statistical distribution of any radiative transfer characteristic $f_Y(Y)$ as specified above ($Y=r,t,\ldots$), provided that conditions valid for the theory presented here are fulfilled? The answer on this question is positive.

To show this, we use the following relationship derived in the probability theory[15]:

$$f_Y(Y)=\left|\frac{\mathrm{d}\tau}{\mathrm{d}Y}\right|f_\tau(\tau). \tag{14}$$

This formula is valid for monotonic (increasing or decreasing) functions $Y(\tau)$. Radiative characteristics in the asymptotic regime ($\tau \to \infty$) satisfy this condition.[13] Therefore, we can use Eq. (14) to relate $f_Y(Y)$ to $f_\tau(\tau)$ in conjunction with Eqs. (3), (4), (6), and (13). In particular, we obtain from Eqs. (6) and (14) (see Fig. 2)

$$f_t(t) = b^{-1} t^{-2} f_\tau(\tau), \tag{15}$$

where $\tau = b^{-1}(t^{-1} - a)$ [see Eq. (6)]. This means that $f_\tau(\tau)$ can be obtained from the measured function $f_t(t)$ as follows:

$$f_\tau(\tau) = b t^2(\tau) f_t(t(\tau)). \tag{16}$$

Functions $T(\xi, \eta)$ and $t_d(\xi)$ are related to t linearly. This means that

$$f_T(T) = f_t(t) K_0^{-1}(\xi) K_0^{-1}(\eta), \qquad f_{t_d}(t_d) = f_t(t) K_0^{-1}(\xi),$$

$$f_T(T) = f_{t_d}(t_d) K_0^{-1}(\eta), \tag{17}$$

and, therefore,

$$f_\tau(\tau) = b t^2 f_{t_d}(t_d) K_0(\xi), \qquad f_\tau(\tau) = b t^2 f_T(T) K_0(\xi) K_0(\eta). \tag{18}$$

We also find that functions $f_R(R)$, $f_{r_d}(r_d)$, and $f_r(r)$ coincide with $f_T(T)$, $f_{t_d}(r_d)$, $f_t(r)$, respectively [see Eqs. (3) and (13)]. Therefore, one concludes that measurements of the statistical distributions of radiative characterististics above and below a scattering nonabsorbing turbid layer can be used to derive the optical thickness distribution using the analytical equations given above. The results are quite general and not restricted to specific types of statistical distributions.

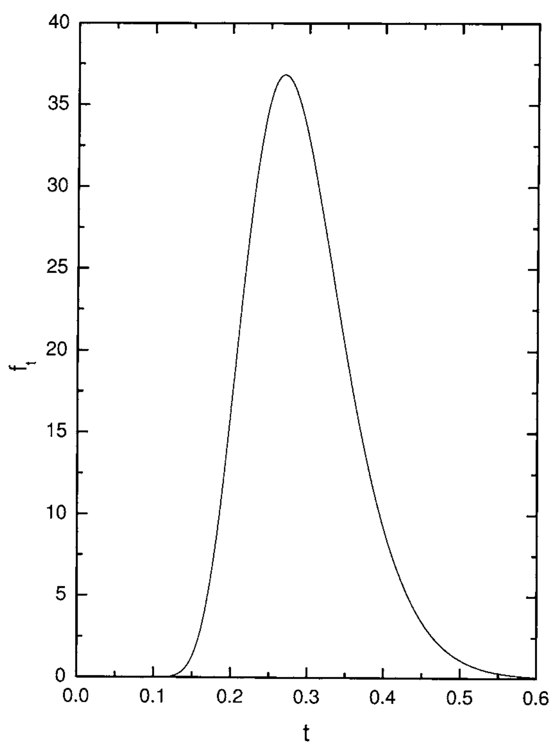

Fig. 2. Statistical distribution of the total light transmittance by a turbid layer having the statistical distribution of τ shown in Fig. 1. Calculations were performed using Eq. (15) at $g=0.85$ and assuming that $\tau = (t^{-1} - a) b^{-1}$ [see Eq. (6)], where $b = 0.75(1-g)$.

3. ACCOUNTING FOR LIGHT ABSORPTION

All substances at all wavelengths absorb some portion of incident light for real-world scattering media. Therefore, it is of importance to study how the effect of light absorption in a turbid layer influences the results given above.

The starting point is the generalized form of Eqs. (3) and (4)[12] for a homogeneous light-scattering plane-parallel layer valid as $\tau \to \infty$ for arbitrary absorption and phase function:

$$R(\xi, \eta, \varphi) = R_\infty(\xi, \eta, \varphi) - T(\xi, \eta) N e^{-k\tau}, \tag{19}$$

$$T(\xi, \eta) = t C^{-2} K(\xi) K(\eta), \tag{20}$$

where

$$t = \frac{M C^2 e^{-k\tau}}{1 - N^2 e^{-2k\tau}} \tag{21}$$

is the global transmittance for an absorbing turbid layer. Functions $R_\infty(\xi, \eta, \varphi)$ and $K(\eta)$ have the same sense as $R_{0\infty}(\xi, \eta, \varphi)$ and $K_0(\eta)$, respectively, but for an absorbing layer. The constant C is defined as

$$C = 2 \int_0^1 K(\eta) \eta \mathrm{d}\eta, \tag{22}$$

which is equal to 1 for nonabsorbing layers [see Eq. (8)]. The procedures to calculate the auxiliary functions $R_\infty(\xi, \eta, \varphi)$ and $K(\eta)$ and constants M, N, k are well known.[11–13] It is of importance for us that these functions and constants do not depend on the optical thickness. They are determined solely by the phase function $p(\theta)$ and the single-scattering albedo ω_0. These characteristics are assumed to be constant inside a turbid layer under study. In particular, it follows as $\omega_0 \to 1$ (Ref. 13): $k \approx [3(1-\omega_0) \times (1-g)]^{1/2}$, $MK(\xi)K(\eta) \approx (1 - \exp(-2y)) K_0(\xi) K_0(\eta)$, $R_\infty = R_{0\infty} \exp(-y R_{0\infty}^{-1} K_0(\xi)\, K_0(\eta)$, $N \approx \exp(-y)$, where $y = 4\sqrt{(1-\omega_0)/3(1-g)}$. Useful approximations for $R_{0\infty}(\xi, \eta, \varphi)$ can be found in Ref. 13.

Equations (19)–(21) give the dependence of reflection and transmission functions with respect to τ in the analytical form. So they can be used to find the derivative as specified in Eq. (14) analytically. As in the nonabsorbing case, we start from the search of the relationship between $f_t(t)$ and $f_\tau(\tau)$.

For this, we express τ via t using Eq. (21). This gives

$$\tau = k^{-1} \ln \left\{ \frac{2tN^2}{MC^2(B(t) - 1)} \right\}, \tag{23}$$

where

$$B(t) = \sqrt{1 + \frac{4N^2 t^2}{M^2 C^4}}. \tag{24}$$

Therefore, we have for the derivative

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{1}{kt} \left\{ 1 - \frac{4N^2 t^2}{B(B-1) M^2 C^4} \right\}, \tag{25}$$

where t is given by Eq. (21), and we omitted arguments. This equation is simplified as $t \to 0$:

$$\frac{d\tau}{dt} = -\frac{1}{kt}. \tag{26}$$

Therefore, introducing

$$\Phi(\tau) = \frac{1}{kt}\left\{1 - \frac{4N^2t^2}{B(B-1)M^2C^4}\right\}, \tag{27}$$

we obtain

$$f_t(t) = \Phi(\tau) f_\tau(\tau). \tag{28}$$

This analytical equation allows one to determine the statistical distribution of the optical thickness from the global transmission measurements [see Eq. (7)] as

$$f_\tau(\tau) = \Phi^{-1}(\tau) f_t(t(\tau)). \tag{29}$$

Using Eqs. (9), (20), and (22), we derive for the diffuse transmittance

$$t_d(\xi) = tC^{-1}K(\xi). \tag{30}$$

This means that

$$f_{t_d}(t_d) = CK^{-1}(\xi) f_t(t). \tag{31}$$

Similarly, one finds

$$f_T(T) = C^2K^{-1}(\xi)K^{-1}(\eta) f_t(t). \tag{32}$$

So we have

$$f_\tau(\tau) = \Phi^{-1}(\tau)C^{-1}K(\xi) f_{t_d}(t_d),$$

$$f_\tau(\tau) = \Phi^{-1}(\tau)C^{-2}K(\xi)K(\eta) f_T(T). \tag{33}$$

It follows that the statistical distributions of the light transmittance under different measurement schemes (e.g., diffused or direct light illumination conditions) are interrelated and governed by the statistical distribution of the optical thickness. Equations (29) and (33) can be used to determine the statistical distribution of the optical thickness from measurements of the transmitted light.

Let us consider the reflectance now. First of all, we write, taking into account Eqs. (10), (11), and (19),

$$r_d(\xi) = r_{d\infty}(\xi) - tC^{-1}K(\xi)N\,e^{-k\tau} \tag{34}$$

and

$$r = r_\infty - tN\,e^{-k\tau} \tag{35}$$

or

$$r = r_\infty - \frac{MNC^2\,e^{-2k\tau}}{1 - N^2\,e^{-2k\tau}}. \tag{36}$$

So we have from Eq. (36)

$$\tau = \frac{1}{2k}\ln\left\{N^2 + \frac{MNC^2}{r_\infty - r}\right\}, \tag{37}$$

$$\frac{d\tau}{dr} = [2k(r_\infty - r)(1 + M^{-1}NC^{-2}(r_\infty - r))]^{-1}. \tag{38}$$

Therefore, introducing

$$Q(r) = [2k(r_\infty - r)(1 + M^{-1}NC^{-2}(r_\infty - r))]^{-1}, \tag{39}$$

we have

$$f_r(r) = Q(r) f_\tau(\tau), \qquad f_\tau(\tau) = Q^{-1}(r(\tau)) f_r(r(\tau)), \tag{40}$$

where $r(\tau)$ is given by Eq. (36). This equation allows us to find the statistical distribution of the optical thickness from measurements of the spherical albedo r. Similar results can be obtained for $f_R(R), f_{r_d}(r_d)$. Namely, we have, following the procedure given above,

$$f_\tau(\tau) = D^{-1}(R) f_R(R), \qquad f_\tau(\tau) = W^{-1}(r_d) f_{r_d}(r_d), \tag{41}$$

where

$$D(R) = [2k(r_\infty - r)(1 + M^{-1}NC^{-2}K(\xi)K(\eta)(r_\infty - r))]^{-1}, \tag{42}$$

$$W(r_d) = [2k(r_\infty - r)(1 + M^{-1}NC^{-1}K(\xi)(r_\infty - r))]^{-1}. \tag{43}$$

Equations (29), (33), (40), and (41) can be used to relate statistical distributions for reflectance and transmittance. Results for the statistical characteristics of light absorption processes such as $a = 1 - r - t$ and $a_d(\xi) = 1 - r_d(\xi) - t_d(\xi)$ can be also easily derived from the equations given above.

4. CONCLUSION

In conclusion, we underline that, as expected, reflection, transmission, and absorption statistical distributions are determined by the statistical distribution of the optical thickness of a random medium under study. All these distributions are interrelated as clearly demonstrated in this work using the analytical approach valid as $\tau \to \infty$ for arbitrary absorption and phase functions.

The statistical distributions derived can be used to find the average values of correspondent characteristics, their higher-order moments (e.g., standard deviations), and relationships between various moments (see, e.g., Ref. 16).

The derivation of statistical distributions of the optical thickness and also correspondent distributions for transmittance from reflectance measurements are of importance in a range of remote-sensing techniques (e.g., in cloud airborne and satellite remote sensing[17,18]).

The derived equations can be used also for the determination of the optical thickness of a homogeneous random layer from reflection or transmission measurements [see, e.g., Eq. (37)]. However, the auxiliary functions and parameters must be known in advance for such a procedure. Simple approximate equations for them [dependent on the pair (ω_0, g)] are given in Refs. 4, 5, 13, and 19.

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